4. Symmetrical and Unsymmetrical Bending

- 4.1 Internal Forces in Beams
- 4.2 Simple Formula for Normal Stresses Due to Pure (or Transverse) Bending
- 4.3 Unsymmetric Bending of Straight Beams
- 4.4 Case of Combined Normal Force and Bending Moments
  - Assumption of Plane Cross Section Remaining Plane
- 4.5 Calculation of Displacements
  - Case of Transverse Bending

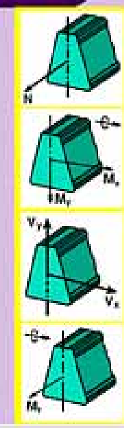
Internal Forces in Beams

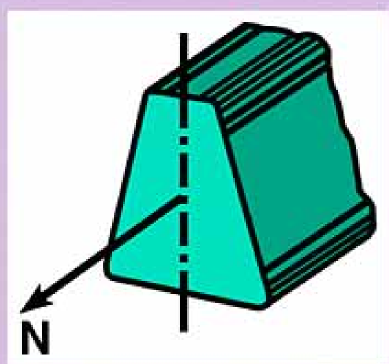
Forces Associated with Normal Stresses

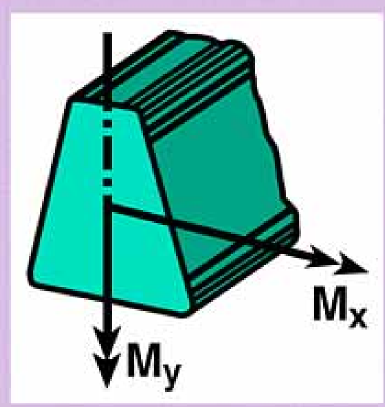
Normal (or axial) force	N
Bending moment (plane yz)	M_x
Bending moment (plane xz)	M_y

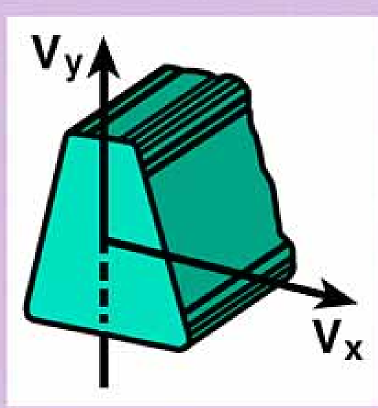
Forces Associated with Shear Stresses

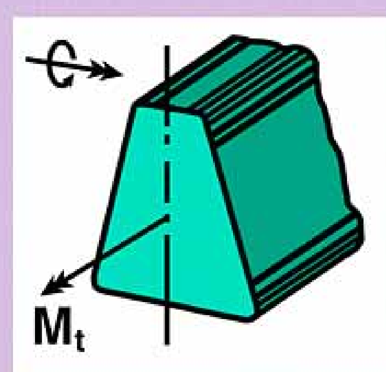
Shearing force	V_x
Shearing force	V_y
Twisting moment (plane xy)	M_t





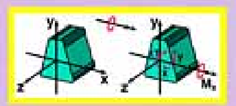






Simple Formula for Normal Stresses Due to Pure (or Transverse) Bending

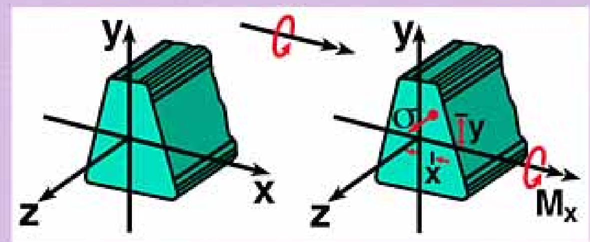
$$\sigma = \frac{M_x\, y}{I_x}$$



Assumptions

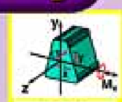
- Plane of loads (and of bending) is perpendicular to neutral axis
- x axis is the neutral axis and is a principal centroidal axis

$$S_x = \int_A y\, dA = 0 \;,\; I_{xy} = \int_A xy\, dA = 0$$

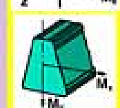


Simple Formula for Normal Stresses Due to Pure (or Transverse) Bending

Internal Forces at the Cross Section



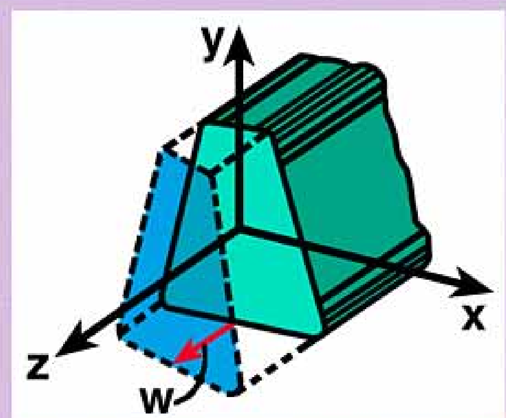
- Normal force N



- Bending moments M_x and M_y about the x and y axes

Kinematic Relations

- Plane cross section before deformation is assumed to remain plane after deformation

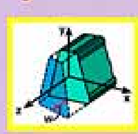


Simple Formula for Normal Stresses Due to Pure (or Transverse) Bending

Kinematic Relations

- Plane cross section before deformation is assumed to remain plane after deformation
- Both the displacement w in the axial direction and the strain ε are linear functions of x and y

$$\varepsilon = \frac{\partial w}{\partial z} = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$



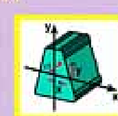
a, b, c are independent of x and y

Simple Formula for Normal Stresses Due to Pure (or Transverse) Bending

Static Relations

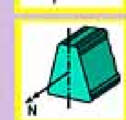
- Sum of internal stresses in the axial direction

$$\int_A \sigma\, dA = N$$



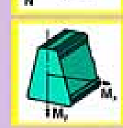
- Sum of moments of internal stresses about y axis

$$\int_A \sigma\, x\, dA = M_y$$



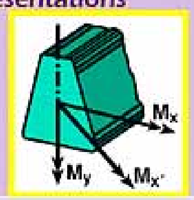
- Sum of moments of internal stresses about x axis

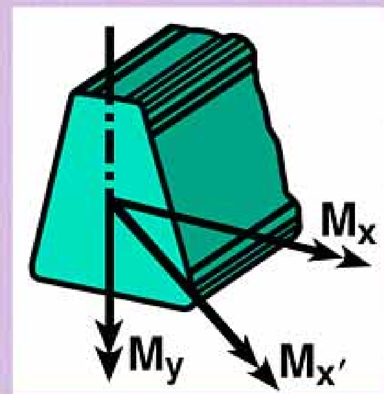
$$\int_A \sigma\, y\, dA = M_x$$



Unsymmetric Bending of Straight Beams

- When the plane of bending is not a principal plane, then
  - Either use a more general formula for the normal stresses (resulting from a bending moment $M_{x'}$), or
  - Decompose the bending moment into components whose vector representations are along principal centroidal axes





Case of Combined Normal Force and Bending Moments

Constitutive Relations

- For linearly elastic material - uniaxial stress state

$$\sigma = E\,\varepsilon$$

$$= E \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$

- From the static relations

$$E \begin{bmatrix} A & S_y & S_x \\ S_y & I_y & I_{xy} \\ S_x & I_{xy} & I_x \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$

Case of Combined Normal Force and Bending Moments

- From the static relations

$$E \begin{bmatrix} A & S_y & S_x \\ S_y & I_y & I_{xy} \\ S_x & I_{xy} & I_x \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$

where

$$\begin{Bmatrix} A \\ S_x \\ S_y \end{Bmatrix} = \int_A \begin{Bmatrix} 1 \\ y \\ x \end{Bmatrix} dA \; ; \quad \begin{Bmatrix} I_x \\ I_y \\ I_{xy} \end{Bmatrix} = \int_A \begin{Bmatrix} y^2 \\ x^2 \\ xy \end{Bmatrix} dA$$

Case of Combined Normal Force and Bending Moments

where

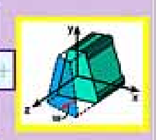
$$\begin{Bmatrix} A \\ S_x \\ S_y \end{Bmatrix} = \int_A \begin{Bmatrix} 1 \\ y \\ x \end{Bmatrix} dA \; ; \quad \begin{Bmatrix} I_x \\ I_y \\ I_{xy} \end{Bmatrix} = \int_A \begin{Bmatrix} y^2 \\ x^2 \\ xy \end{Bmatrix} dA$$

or

$$\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} A & S_y & S_x \\ S_y & I_y & I_{xy} \\ S_x & I_{xy} & I_x \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$

Case of Combined Normal Force and Bending Moments

$$\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} A & S_y & S_x \\ S_y & I_y & I_{xy} \\ S_x & I_{xy} & I_x \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$



- The general formula for normal stresses in terms of the normal force and bending moments:

$$\sigma = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} A & S_y & S_x \\ S_y & I_y & I_{xy} \\ S_x & I_{xy} & I_x \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$

Case of Combined Normal Force and Bending Moments

Simplifications

- If x and y are centroidal axes, then
  $S_x = S_y = 0$

$$\sigma = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} A & 0 & 0 \\ 0 & I_y & I_{xy} \\ 0 & I_{xy} & I_x \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$

- If x and y are centroidal principal axes, then $S_x = S_y = 0$, $I_{xy} = 0$

Case of Combined Normal Force and Bending Moments

- If x and y are centroidal principal axes, then $S_x = S_y = 0$, $I_{xy} = 0$

$$\sigma = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} \frac{1}{A} & \cdot & \cdot \\ \cdot & \frac{1}{I_y} & \cdot \\ \cdot & \cdot & \frac{1}{I_x} \end{bmatrix} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$

$$\sigma = \frac{N}{A} + \frac{M_y}{I_y} x + \frac{M_x}{I_x} y$$

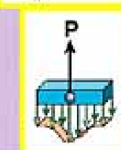
Case of Combined Normal Force and Bending Moments

$$\sigma = \frac{N}{A} + \frac{M_y}{I_y} x + \frac{M_x}{I_x} y$$

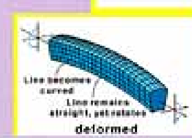
Neutral Axis
Is the axis at which the normal stress $\sigma = 0$

Case of Combined Normal Force and Bending Moments

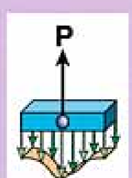
- Assumption of plane cross section before deformation remains plane
  - Is accurate for:
    - axial load (away from point of application of load)
    - pure bending
  - Is approximate for transverse bending
- Assumption of undeformable cross sections

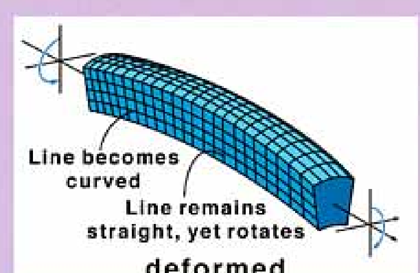


$$\varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$$



- Only approximate for bending



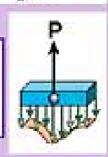


Case of Combined Normal Force and Bending Moments

- Assumption of undeformable cross sections

$$\varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$$

- Only approximate for bending



$$\left(\varepsilon_x\ ,\ \varepsilon_y\ ,\ \gamma_{xy}\right) << \varepsilon_z$$

- More approximate than for beams under axial loading

$$\varepsilon_x = \varepsilon_y = -\upsilon\varepsilon_z$$

Calculations of Displacements

Rotation, Curvature and Axial Strain

- For the case of a single bending moment Mx

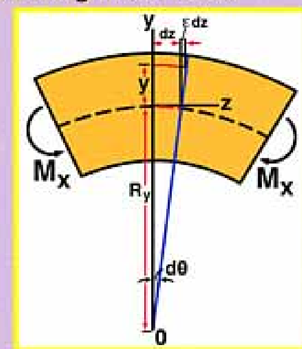
$$d\theta = \frac{dz}{R_y} = \frac{\varepsilon\, dz}{y}$$

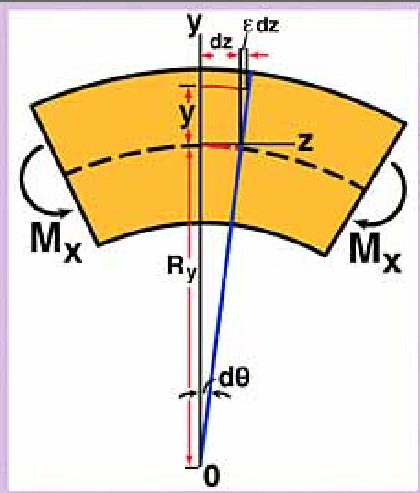
$$\frac{1}{R_y} = \frac{d\theta}{dz} = \frac{\varepsilon}{y}$$

$$\simeq -\frac{d^2v}{dz^2}$$

or

$$\varepsilon \simeq -y\,\frac{d^2v}{dz^2}$$





Calculations of Displacements

- Analogously, for a bending moment M_y

$$\varepsilon \simeq -x\,\frac{d^2u}{dz^2}$$

- And, for an axial force N $\quad \varepsilon = \varepsilon_o$

- For the case of combined axial force, bending moment M_x and bending moment M_y

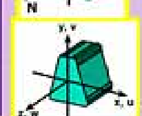
$$\varepsilon = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} \varepsilon_o \\ -\dfrac{d^2u}{dz^2} \\ -\dfrac{d^2v}{dz^2} \end{Bmatrix}$$

Calculations of Displacements

Displacement equations from which

$$\varepsilon = \frac{dw}{dz} = \frac{\sigma}{E}$$

$$\begin{Bmatrix} \varepsilon_o \\ -\dfrac{d^2u}{dz^2} \\ -\dfrac{d^2v}{dz^2} \end{Bmatrix} = \frac{1}{E} \begin{vmatrix} A & S_y & S_x \\ S_y & I_y & I_{xy} \\ S_x & I_{xy} & I_x \end{vmatrix}^{-1} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$



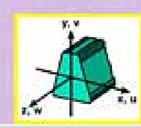
Calculations of Displacements

$$\begin{Bmatrix} \varepsilon_o \\ -\dfrac{d^2u}{dz^2} \\ -\dfrac{d^2v}{dz^2} \end{Bmatrix} = \frac{1}{E} \begin{vmatrix} A & S_y & S_x \\ S_y & I_y & I_{xy} \\ S_x & I_{xy} & I_x \end{vmatrix}^{-1} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$

Simplifications

- If x and y are centroidal axes, and N is absent, then

$$\begin{Bmatrix} \dfrac{d^2u}{dz^2} \\ \dfrac{d^2v}{dz^2} \end{Bmatrix} = \frac{-1}{E} \begin{vmatrix} I_y & I_{xy} \\ I_{xy} & I_x \end{vmatrix}^{-1} \begin{Bmatrix} M_y \\ M_x \end{Bmatrix}$$



Calculations of Displacements

Simplifications

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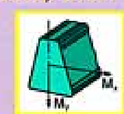
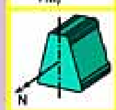
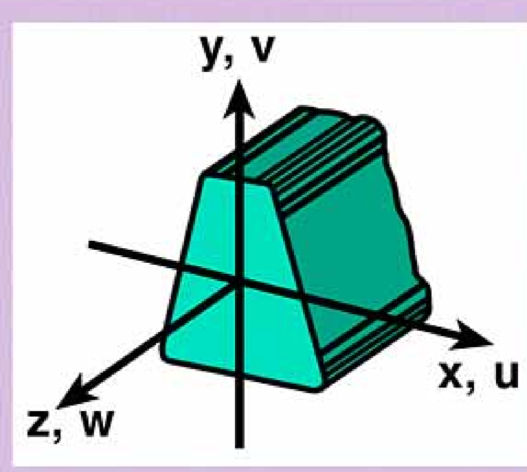
$$\begin{Bmatrix} \dfrac{d^2u}{dz^2} \\ \dfrac{d^2v}{dz^2} \end{Bmatrix} = \frac{-1}{E} \begin{vmatrix} I_y & I_{xy} \\ I_{xy} & I_x \end{vmatrix}^{-1} \begin{Bmatrix} M_y \\ M_x \end{Bmatrix}$$





- If x and y are centroidal principal axes, and N is absent, then

$$\begin{Bmatrix} \dfrac{d^2u}{dz^2} \\ \dfrac{d^2v}{dz^2} \end{Bmatrix} = \frac{-1}{E} \begin{Bmatrix} \dfrac{M_y}{I_y} \\ \dfrac{M_x}{I_x} \end{Bmatrix}$$



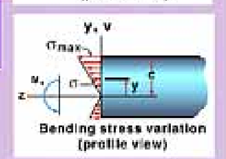
Calculations of Displacements

Case of Transverse Bending

- Governing equation for the elementary theory of beams:

$$\frac{d^2}{dz^2}\left(EI_x \frac{d^2v}{dz^2}\right) = p_y$$

- For uniform beams



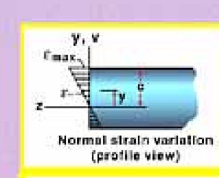
Calculations of Displacements

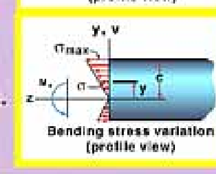
Case of Transverse Bending

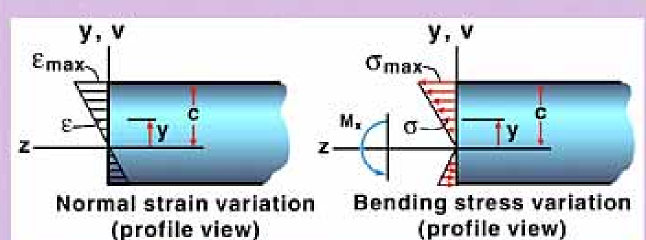
- For uniform beams

$$EI\frac{d^4v}{dz^4} = p$$

where subscripts x and y have been dropped for convenience.







Calculations of Displacements

Case of Transverse Bending

- Successive integration of the differential equation

Transverse shear

$$EI\frac{d^3v}{dz^3} = \int_0^z p\,dz + c_1 = -\mathbf{V}$$

Calculations of Displacements

Case of Transverse Bending

Bending Moment

$$EI\frac{d^2v}{dz^2} = \int_0^z\int_0^z p\,dz\,dz + c_1z + c_2$$
$$= -M$$

Calculations of Displacements

Case of Transverse Bending

Slope

$$EI\frac{dv}{dz} = \int_0^z\int_0^z\int_0^z p\,dz\,dz\,dz + \frac{1}{2}c_1z^2 + c_2z + c_3$$

Calculations of Displacements

Case of Transverse Bending

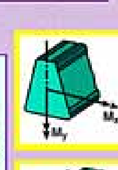
Transverse Displacement

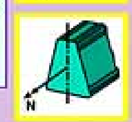
$$EI\;v = \int_0^z\int_0^z\int_0^z\int_0^z p\,dz\,dz\,dz\,dz + \frac{1}{6}c_1z^3 + \frac{1}{2}c_2z^2 + c_3z + c_4$$

Calculations of Displacements

- Case of combined $\boxed{N, M_x, M_y}$

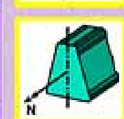
$$\sigma = \begin{bmatrix}1 & x & y\end{bmatrix}\begin{bmatrix}A & S_y & S_x\\ S_y & I_x & I_{xy}\\ S_x & I_{xy} & I_x\end{bmatrix}^{-1}\begin{Bmatrix}N\\ M_y\\ M_x\end{Bmatrix}$$

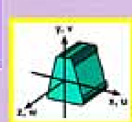




Calculations of Displacements

- Displacement equations





$$\begin{Bmatrix}\varepsilon_o\\ -\frac{d^2u}{dz^2}\\ -\frac{d^2v}{dz^2}\end{Bmatrix} = \frac{1}{E}\begin{bmatrix}A & S_y & S_x\\ S_y & I_y & I_{xy}\\ S_x & I_{xy} & I_x\end{bmatrix}^{-1}\begin{Bmatrix}N\\ M_y\\ M_x\end{Bmatrix}$$

Calculations of Displacements

- If x,y are centroidal principal axes

$$\frac{dw}{dz} = \frac{N}{EA}$$

$$-\frac{d^2u}{dz^2} = \frac{M_y}{EI_y} \qquad -\frac{d^2v}{dz^2} = \frac{M_x}{EI_x}$$

w is the displacement of the axis of the beam in the z direction